Election Systems and Fairness Criteria in Voting: Lesson and Activity Packet

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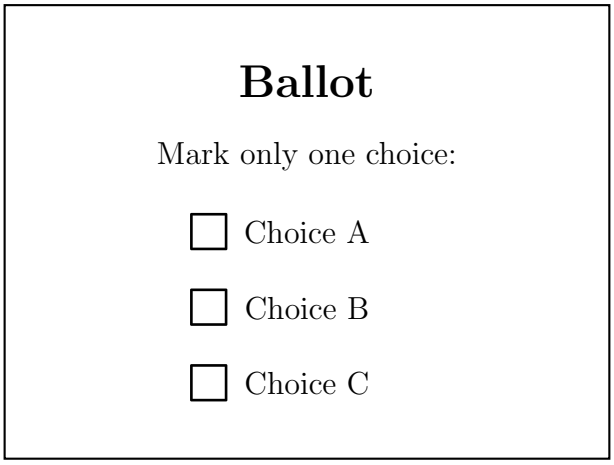
April 22, 2016

An *election* is just a kind of decision-making process (a way of using mathematics!).

For U.S. presidential (general) elections, we use *ballots* of the following kind:



We are also familiar with the *election system* used in deciding the winner of the U.S. presidential elections[1]:

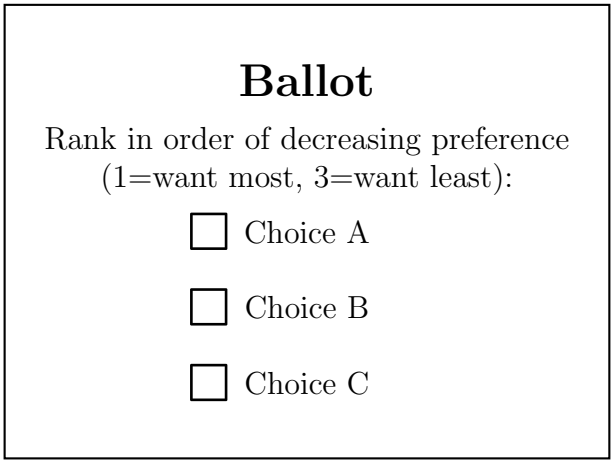
Election System 1 (*Plurality System*)

The candidate with the most votes is declared the winner.

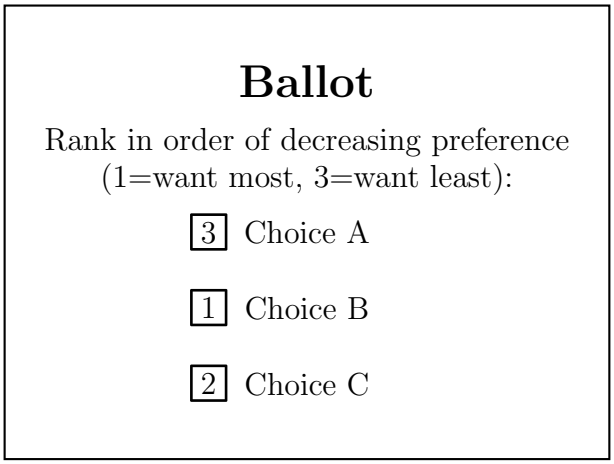
But this kind of method might leave out critical information about how each voter ranks the *entire* list of candidates.

[1] In reality, the *voters* in the system used in U.S. presidential elections are not the citizens themselves, but the 538 members (called "electors") of the Electoral College, who vote on the citizens' behalf.

From now on, instead of the "plurality ballots" from the previous page, we will use these "preference ballots", with just a change to the instructions:



Tallying preference ballots requires keeping track of more information. This following sample ballot can be represented as $\begin{matrix} B \\ C \\ A \end{matrix}$:



Let's say we had ten ballots:

voter:	voter 1	voter 2	voter 3	voter 4	voter 5	voter 6	voter 7	voter 8	voter 9	voter 10
1st choice:	B	C	B	C	A	C	C	B	A	C
2nd choice:	A	A	A	B	B	A	A	A	B	B
3rd choice:	C	B	C	A	C	B	B	C	C	A

We can summarize all of these votes in a *preference table*:

how many voters chose this ranking:	(sum is the total # of voters)
1st choice:	
2nd choice:	
3rd choice:	

Group Exercise 1

Consider the following collection of ballots:

voter 1	*voter 2*	*voter 3*	*voter 4*	*voter 5*	*voter 6*	*voter 7*	*voter 8*	*voter 9*	*voter 10*
A	*A*	*B*	*B*	*A*	*A*	*C*	*C*	*C*	*B*
B	*C*	*D*	*D*	*C*	*B*	*B*	*B*	*B*	*D*
C	*B*	*C*	*C*	*B*	*C*	*D*	*D*	*D*	*C*
D	*D*	*A*	*A*	*D*	*D*	*A*	*A*	*A*	*A*

voter 11	*voter 12*	*voter 13*	*voter 14*	*voter 15*	*voter 16*	*voter 17*	*voter 18*	*voter 19*	*voter 20*	*voter 21*
A	*B*	*C*	*C*	*A*	*B*	*C*	*A*	*B*	*B*	*A*
C	*D*	*B*	*B*	*C*	*D*	*B*	*C*	*D*	*D*	*B*
B	*C*	*D*	*D*	*B*	*C*	*D*	*B*	*C*	*C*	*C*
D	*A*	*A*	*A*	*D*	*A*	*A*	*D*	*A*	*A*	*D*

Create a preference table based on this collection of ballots.

Plurality voting can still be done using preference ballots (we only look at the choice ranked "1" in that case—that is, the top row of the preference table).

Election System 2 (*Plurality System (for Preference Ballots)*)

The candidate with the most ***first-place*** *votes is declared the winner.*

Individual Exercise 2

Using the preference table you created, determine which candidate is the winner, under the Plurality System, of the election in Exercise 1.

The *majority* of anything (*e.g.*, votes, voters, cows in a field) is defined as 50% plus one (rounded down to the nearest whole number).

Group Exercise 3

How many votes constitutes a majority in Exercise 1? Did the winner of the election under the Plurality System receive a majority of first-place votes?

Group Exercise 4

If there ***had*** *been a candidate who had received a majority of first-place votes, would that candidate have won under the Plurality System?*

A *fairness criterion* is a way to test if an **election system** is inherently fair—that is, if the candidate it chooses actually reflects the will of the voters.

Fairness Criterion 1 (*Majority Criterion*)

If any candidate receives a majority of first-place votes, then that candidate is called the majority winner*, and the election system should declare that candidate the winner.*

Group Exercise 5

Does there ***always exist*** *a majority winner?*[a] *[Hint: Was there a majority winner in Exercise 1?]*

[a] "Existence" [of solutions/winners/objects/etc.] is a very important notion in mathematics!

Group Exercise 6

Can there ever be more than one majority winner? In other words, is a majority winner ***unique****?*[a]

[a] "Uniqueness" is also a very important notion in mathematics!

A related fairness criterion...

Fairness Criterion 2 (*Majority-Loser Criterion*)

If any candidate receives a majority of last-place votes, then that candidate is called the majority loser*, and the election system should* ***not*** *declare that candidate the winner.*

Individual Exercise 7

- *Was there a majority loser in Exercise 1?*
- *If so, which candidate was that?*
- *Who was the winner of the election under the Plurality System in Exercise 1?*
- *Does the Plurality System satisfy the Majority-Loser Criterion?*

An example that **disproves** an assertion is called a *counterexample*, and is another very important mathematical concept!

In order to determine the most "fair" winner of the election from Exercise 1, we will use more of the information from the preference table.

Preference tables tell us how voters rank each candidate relative to *any* of the others, via *head-to-head comparisons*, like in a "round-robin"-style sporting tournament.

Let's examine the head-to-head comparisons with Candidate A from Exercise 1:

A-vs-B

3	5	7	6
A	A	B	C
B	C	D	B
C	B	C	D
D	D	A	A

A-vs-C

3	5	7	6
A	A	B	C
B	C	D	B
C	B	C	D
D	D	A	A

A-vs-D

3	5	7	6
A	A	B	C
B	C	D	B
C	B	C	D
D	D	A	A

Group Exercise 8

Find the winners of each head-to-head comparison with Candidate C from Example 1:

C-vs-A

3	*5*	*7*	*6*
A	*A*	*B*	*C*
B	*C*	*D*	*B*
C	*B*	*C*	*D*
D	*D*	*A*	*A*

C-vs-B

3	*5*	*7*	*6*
A	*A*	*B*	*C*
B	*C*	*D*	*B*
C	*B*	*C*	*D*
D	*D*	*A*	*A*

C-vs-D

3	*5*	*7*	*6*
A	*A*	*B*	*C*
B	*C*	*D*	*B*
C	*B*	*C*	*D*
D	*D*	*A*	*A*

You found in Exercise 8 that Candidate C is the winner of *every* head-to-head comparison with the other candidates. This suggests another fairness criterion...

Fairness Criterion 3 (*Condorcet Criterion*)

If a candidate wins in ***every*** *head-to-head comparison with the other candidates, then that candidate is called the* Condorcet winner*, and the election system should declare that candidate the winner.*[a]

[a] This criterion was devised by and is named after the Marquis de Condorcet, one of the two pioneers of the field of voting theory, who lived from 1743–1794.

Group Exercise 9

Does the Plurality System satisfy the Condorcet Criterion? How do you know?

Group Exercise 10

Is a Condorcet winner ***unique****? (That is, can there ever be more than one Condorcet winner in an election?) Why or why not?*

The existence problem: Not every election **has** a Condorcet winner! Example:

1	1	1
A	C	B
B	A	C
C	B	A

The head-to-head comparisons:

A-to-B

1	1	1
A	C	B
B	A	C
C	B	A

A-to-C

1	1	1
A	C	B
B	A	C
C	B	A

B-to-C

1	1	1
A	C	B
B	A	C
C	B	A

Election System 3 (*Copeland's Method*)

Consider all pairwise (head-to-head) comparisons, and award each candidate one point for a win, and one half-point for a tie. The total number of points awarded to a candidate is called that candidate's Copeland score*. The candidate with the highest Copeland score is declared the winner.*

To compute Copeland scores, we will use a more convenient way of displaying the outcomes of the head-to-head comparisons. An example:

Recall that our preference table from Example 1 was:

3	5	7	6
A	A	B	C
B	C	D	B
C	B	C	D
D	D	A	A

Let's create a table whose entries show the outcomes of head-to-head comparisons between rows and columns:

	A	B	C	D
A				
B				
C				
D				

. . . and we can use this table to find the Copeland winner.

Let's look at a **different** example with the Copeland Method.

There are four candidates running to be the president of our class.

- Candidate A: _____
- Candidate B: _____
- Candidate C: _____
- Candidate D: _____

We held the election, and the preference table is:

5	5	6	4
D	A	C	B
A	C	B	D
C	B	D	A
B	D	A	C

Group Exercise 11

Build the comparison table for this election.

	A	B	C	D
A				
B				
C				
D				

Group Exercise 12

The winner of the election is: _____.

After the election is complete but before the results are made public, Candidate D announces that she has withdrawn from the race.

This reduces the comparison table to:

	A	B	C
A			
B			
C			

Group Exercise 13

Who is the winner now?

Fairness Criterion 4 (*Independence of Irrelevant Alternatives*)

If a non-winning candidate is removed from the ballot, it should not change the winner of the election.

Individual Exercise 14

Does the Copeland Method satisfy the Independence of Irrelevant Alternatives Criterion?

The example in Exercises 12 and 13 proves that the Copeland Method **does not** satisfy the Independence of Irrelevant Alternatives Criterion.

. . . So, to recap: For our data in Exercise 1, the Plurality winner was Candidate A, and the Copeland winner was Candidate C.

None of our methods counted the *margins of victory*. For example:

- B and C both beat A by 8–13
- B beats D in a landslide 21–0
- C beats D by 14–7
- C beats B by the skin of her teeth(!), 11–10

Let's look at another method that **does** consider the margin of victory.

Election System 4 (*Borda Count*)

For each ballot cast, award zero points to the candidate ranked last, one point for the candidate ranked next-to-last, and so forth, awarding $N-1$ points to the candidate ranked first (where N is the total number of candidates). For each candidate, the sum of his or her points from each ballot is called that candidate's Borda score.

The candidate with the highest Borda score is declared the winner.[a]

[a]This method was devised by and named after Jean-Charles de Borda, the other of the two pioneers of voting theory, who lived from 1733–1799.

For the election in Exercise 1:

3	5	7	6	
A	A	B	C	N-1 = 3
B	C	D	B	N-2 = 2
C	B	C	D	N-3 = 1
D	D	A	A	N-4 = 0

The Borda score of Candidate A is:

Group Exercise 15

Compute the Borda scores of Candidates B, C, and D using the data above. Who is the winner of the election using the Borda Count system?

Group Exercise: Candy Ranking

Individual Exercise 16

Fill out the ballot indicating your candy-ranking (ballot is on a separate piece of paper).

Group Exercise 17

Make a preference table for the votes in your group.

Group Exercise 18

Compute the winner of your group's election using the Plurality System.

Group Exercise 19

Compute the winner of your group's election using the Copeland Method.

Group Exercise 20

Compute the winner of your group's election using the Borda Count.

Group Exercise 21

Compute the winner of your group's election using the Instant Runoff System.

Recap

We learned about the following **election systems**:

- Plurality System
  - U.S. general presidential elections
- Copeland Method
  - Wikimedia Commons (runs Wikipedia) Board of Trustee elections
  - World Cup soccer finals
- Borda Count
  - Icelandic parliamentary elections
  - Football's Heismann trophy
- Instant-Runoff
  - Australian parliamentary elections
  - Minneapolis, San Francisco mayoral and city council elections
  - Academy Award for "Best Picture"

We learned about the following **fairness criteria**:

- Majority Criterion
- Majority-Loser Criterion
- Condorcet Criterion
- Independence of Irrelevant Alternatives Criterion

We learned the mathematical notions of:

- **Existence** [of solutions/winners/objects/etc.]
- **Uniqueness** [of solutions/winners/objects/etc.]
- **Counterexample** to disprove an assertion
- **Lots of decision-making strategies and some basic notions of game theory!**

Optional Homework / Further Reading

For you to think about...

1. You know the Condorcet Criterion; there is also the notion of a Condorcet Loser Criterion. Without looking it up first, try to formulate a statement of this criterion.

2. Does the Instant Runoff system satisfy the Independence of Irrelevant Alternatives Criterion?

3. Does the Borda Count system satisfy the Condorcet Criterion? Does it satisfy the Condorcet Loser Criterion from Problem 1?

4. Is (a/the) winner in a Borda Count election unique? How do you know?

Further Reading:

- Wikipedia has a table in its article entitled "Voting Systems", which compares 14 different systems (including the four we studied) and 20 different fairness criteria. Find the article and the table here: `https://en.wikipedia.org/wiki/Voting_system`

- The six-minute YouTube video by CGP Gray entitled "The Problem with First-Past-the-Post Voting Explained", at `https://www.youtube.com/watch?v=s7tWHJfhiyo`, explains how the Plurality System works (or doesn't work) in countries with political parties